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Application of ZZ Transform Method for Newton's Law of Cooling

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Abstract

ZZ Transform Method was introduced by Zain Ul Abadin Zafar [1] in 2016. In this paper, I have applied this new integral transform method for finding solutions to the problems occured on a principle Newton's Law of Cooling, which is an application of first order and first degree ordinary linear differential equation. ZZ Transform Method is similar to the other transforms such as Laplace and Sumudu Transforms. The advantagae of ZZ Transform Method is that it solves a differential equation with variable coefficients.

Keywords:

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1. Introduction

In science and engineering Differential equations plays an important role. In order to solve the certain differential equations, the integral transforms were extensively used. The importance of an Integral Transforms is that they provide powerful operational methods for solving initial value problems. Mostly we use Laplace Transform technique for solving differential equations. ZZ Transform Method is the new integral transform method which is introduced by Zain Ul Abadin Zafar [1], is very much useful to solve a differential equation with variable coefficients. Here in this paper ZZ transform method is used for solving the problems on Newtons Law of Cooling.

1.1 Defintion of ZZ Transform

Let f(t) be a function defined for all t≥0. The ZZ transform of f(t) is the function Z(v,s) defined by $Z(v,s) = H\{f(t)\} = s\int_0^\infty f(vt)e^{-st}dt$ __________(1.1.1) provided the integral exists. Equation (1.1.1) can be written as $Z(v,s) = H\{f(t)\} = \frac{s}{v}\int_0^\infty f(t)e^{\frac{-st}{v}}dt$ _____(1.1.2).

1.2 Existance of ZZ Transform Method

If f(t) is sectionally continuous in every finite interval 0≤t≤k and of exponential order ρ for t>k, then its ZZ transform Z(v,s) exists for al s>ρ, v>ρ.

1.3 ZZ Transform of Some Required Functions

(i) Let $f(t)=e^{at}$, where $t\geq 0$ and a is a constant, then $H\left\{e^{at}\right\}=\dfrac{s}{s-va}$ ______(1.3.1)

(ii) Let $f(t)=\begin{cases}1 & t>0\\ 0 & t\leq 0\end{cases}$ then $H\{1\}=1$ _____(1.3.2)

(iii) Let $f(t)=t^n$, where $n\in N$, then $H\left\{t^n\right\}=n!\dfrac{v^n}{s^n}$ ________(1.3.3)

1.4 Linear Property

If a, b are constants and f(t) and g(t) are functions, then

$$H\{a\,f(t)+b\,g(t)\}=a\,H\{f(t)\}+b\,H\{f(t)\} \quad \text{\_\_\_\_\_\_(1.4.1)}$$

1.5 ZZ Transform for Derivatives

Let H{f(t)}=Z(v,s), then $H\left\{f^n(t)\right\}=\dfrac{s^n}{v^n}Z(v,s)-\sum_{k=0}^{n-1}\dfrac{s^{n-1}}{v^{n-k}}f^k(0)$ ________(1.5.1)

2. Newton's Law of Cooling

Newton's law of cooling states that the rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of its surrounding medium.

Let θ be the temperature of the body at any time t, θ_0 be the temperature of its surrounding medium. Then by Newton's law of cooling we have

$\dfrac{d\theta}{dt}\propto[\theta(t)-\theta_0(t)]\Rightarrow\dfrac{d\theta}{dt}=-\mu[\theta(t)-\theta_0(t)]$___(2.1), where k is proportionality constant (k>0).

3. Application of ZZ transform to a problem on Newton's Law of Cooling and Analysis

3.1 Problem: A body temperature is comedown from 80^0C to 60^0C in 20 minutes, when it was placed at which the surrounding air temperature is at 40^0C. Then

(i) What will be the body temperature after 40 minutes?

(ii) When will be the temperature is 55^0C?

Solution: Given that, the surrounding temperature of the air is $\theta_0=40^0C$, initial temperature i.e. at time $t=0,\ \theta(0)=80^0C$ and after time $t=20\,\text{min},\ \theta(20)=60^0C$.

From Newton's Law of cooling (equation 2.1), we have

$$\frac{d\theta}{dt}\propto[\theta(t)-\theta_0(t)]$$

$$\Rightarrow\frac{d\theta}{dt}=-\mu[\theta(t)-\theta_0(t)] \qquad where\ \mu\ is\ a\ constant.$$

$$\Rightarrow\frac{d\theta}{dt}=-\mu\,\theta(t)+\mu\theta_0(t)$$

$$\Rightarrow\frac{d\theta}{dt}+\mu\,\theta(t)=\mu\theta_0(t) \quad or \quad \theta^1(t)+\mu\theta(t)=40\mu \qquad \text{\_\_\_\_\_\_\_\_\_\_\_(3.1.1)}$$

Taking ZZ Transform on both sides of (3.1.1), we get $H\left\{\theta^1(t)+\mu\,\theta(t)\right\}=H\{40k\}$

Using the linear property, we can write the above equation as

$$H\left\{\theta^1(t)\right\}+\mu H\{\theta(t)\}=40\,\mu\,H\{1\}$$

Now by using the property of ZZ transform of derivative,

$$\left[\frac{s}{v}H\{\theta(t)\}-\frac{s}{v}\theta(0)\right]+\mu H\{\theta(t)\}=40\mu \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_(3.1.2)}$$

Since at time t=0, given that $\theta(0)=80^0C$, substituting this in equation (3.1.2), we have

$$\left[\frac{s}{v}+\mu\right]-\frac{s}{v}80=40\mu$$

$$\Rightarrow\left(\frac{\mu v+s}{v}\right)H\{\theta(t)\}=40\mu+\frac{80s}{v}$$

$$H\{\theta(t)\}=\left(\frac{40\mu v+80s}{v}\right).\left(\frac{v}{\mu v+s}\right)=40\left(\frac{\mu v+2s}{v}\right).\left(\frac{v}{\mu v+s}\right)=40\left(\frac{(\mu v+s)+s}{\mu v+s}\right)$$

$$=40\left(1+\frac{s}{\mu v+s}\right) \qquad \text{\_\_\_\_\_\_\_\_\_(3.2.3)}$$

Applying inverse ZZ Transform on both side of equation (3.2.3), then we get

$$\Rightarrow H^{-1}\left[H\{\theta(t)\}\right]=H^{-1}\left\{40\left[1+\frac{s}{\mu v+s}\right]\right\}$$

Since it satisfies the linear property,

$$\Rightarrow\theta(t)=40\left[H^{-1}\{1\}+H^{-1}\left\{\frac{s}{\mu v+s}\right\}\right] \quad \left(\because H^{-1}\left\{\frac{s}{\mu v+s}\right\}=e^{-\mu t}\right)$$

$$=40\left[1+e^{-\mu t}\right] \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(3.2.4)}$$

Using the condition at t=20min,θ(20)=60^0C, from (3) we have

$$60=40+40e^{-20k}\Rightarrow 20=40e^{-20k}\Rightarrow\frac{1}{2}=e^{-20k}\Rightarrow e^{-k}=\left(\frac{1}{2}\right)^{1/20} \qquad \text{\_\_\_\_\_\_\_\_(3.2.5)}$$

(i) To find the temperature after time t=40min, from (3.2.5) we have $\theta(t)=40+40e^{-40k}$

$$\theta(t)=40+40\left(e^{-k}\right)^{40}=40+40\left(\frac{1}{2}\right)^{\frac{40}{20}}=40+40\left(\frac{1}{2}\right)^{\frac{1}{2}}=50^0C.$$

Therefore, after 40 minutes of time the temperature of the body is 50^0C.

(ii) Now to find the time required to become the temperature of the body is 55^0C, again from (3), we get

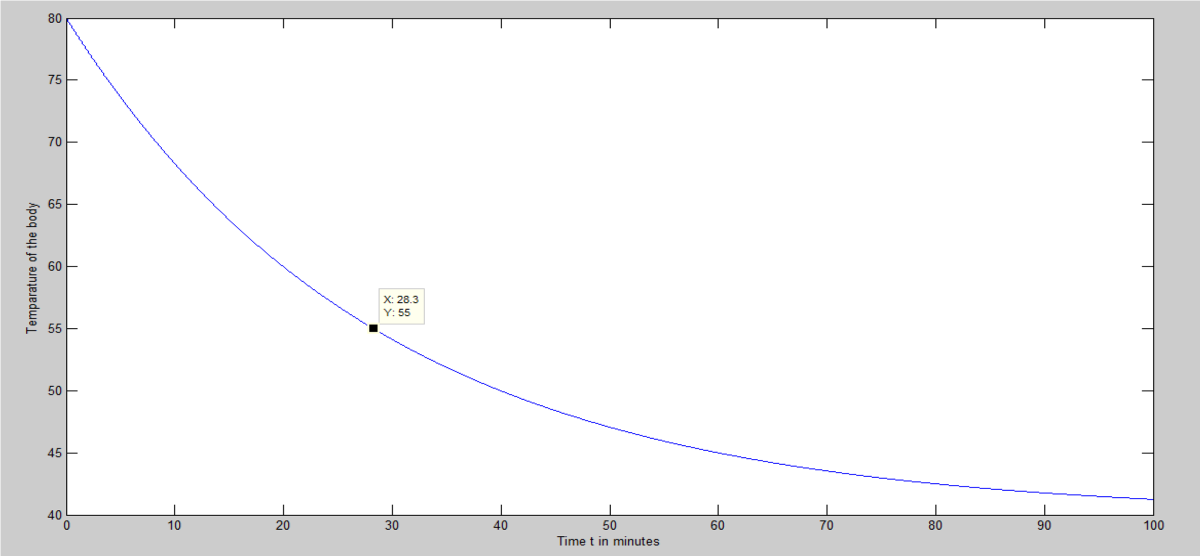
$$55=40+40e^{-kt}\Rightarrow 15=40(e^{-k})^t\Rightarrow\frac{15}{40}=(e^{-k})^t \qquad \text{\_\_\_\_(3.2.6)}$$

Substituting (3.2.5) in (3.2.6), we get $\frac{15}{40}=(\frac{1}{2})^{t/20}$ .Taking natural logarithms on both side's of this equation, then we have

$$\log_e\left(\frac{15}{40}\right)=\frac{t}{20}\log_e\left(\frac{1}{2}\right)\Rightarrow t=\frac{20\log_e\left(\frac{15}{40}\right)}{\log_e\left(1/2\right)}\min=28.3\min utes$$

Therefore after 28.3 minutes of time, the body temperature is 55^0C.

The graph is drawn by taking the time on X-axis and the temperature of the body is on Y-axis.



4. Conclusion

We can apply ZZ Transform Method for solving ordinary differential equation, which is obtained based on the principle Newton's Law of Cooling.

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